Effects of Nonlinearity on Anderson Localization

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While the study of waves in disordered media, including effects such as Anderson localization, and the study of nonlinear systems, including effects such as soliton propagation, have each received considerable attention, the combined field, nonlinear wave propagation in disordered media, is relatively new. A fundamental question is: Does nonlinearity weaken Anderson localization? In this paper we report the results of an experiment, which can provide conditions filling a gap in the current theory, showing that states remain Anderson localized in the presence of nonlinearity.

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Considerable progress has been made in understanding the wave mechanical properties of disordered systems, through extensive theoretical and experimental studies of Anderson localization, coherent backscatter, etc. [1]. Similarly, there has been considerable theoretical and experimental research with nonlinear systems, including the phenomena of soliton propagation, chaos, and other novel effects [2]. Both of the fields of disorder and nonlinearity are individually of practical importance as well as of fundamental scientific interest, and the two fields have undergone rapid development independently, with relatively little overlap. However, the combined effects of disorder and nonlinearity occur frequently in nature, as in many-electron mesoscopic devices, high-intensity optical systems, biological and polymer systems, etc., and there should be even more remarkable effects of fundamental interest. For example, a fundamental question is: Does nonlinearity weaken Anderson localization? Recently there has been significant theoretical progress regarding this question [3-10], but there have been few direct experimental studies. In this paper we report the results of an experiment in a one-dimensional system which determines the effects of nonlinearity on Anderson localization. Before stating the results, the question itself must be clarified.

While powerful theoretical tools have been developed for studying the consequences of disorder, the added effect of nonlinearity greatly complicates the problem. With respect to the question of nonlinearity weakening Anderson localization, there are about eight theoretical papers (or sequences of papers) in the literature [3-10], and roughly half of these papers predict that nonlinearity will weaken Anderson localization [3-6], and the others predict that it will not [6-10]. While this statement seems to imply a controversy, the theories are in fact not contradictory, because, as is common in nonlinear problems, the question does not have a unique answer. For example, one may pose a problem as involving a pulse (or soliton) [3,4] or as involving an extended, single frequency wave [6,7] propagating through disordered scatterers; in a linear system all results would be equivalent, simply related by a Fourier transform. However, in a nonlinear system the different ways of posing the problem are no longer equivalent. Even in the case of an extended single frequency wave there is more than one answer concerning the effects of nonlinearity on Anderson localization. One considers the problem of how the transmission of a continuous wave through a disordered one-dimensional region varies with the length of the region. If the incident wave amplitude is held constant, then the transmitted power is not necessarily unique, although it has been shown [7] that the transmission may still decay exponentially with length as for the linear disordered system. On the other hand, if the output amplitude is held constant, then one may obtain the unique result that the exponential decay is replaced with a power-law decay [6]. However, holding the output amplitude constant is not a typical way of defining a transmission measurement; usually one holds the input amplitude constant, and in this case, nonuniqueness notwithstanding, the Anderson localization is not weakened.

Some of the theory papers [8-10] develop rigorous theorems addressing the effects of nonlinearity on Anderson localization, such as, for example, the paper by Frolich, Spencer, and Wayne (FSW) [8]. This paper considers the existence of exponentially localized solutions of a Hamiltonian with a nonlinear term, and the result is that under general conditions Anderson localization is still present when there is nonlinearity. An interesting problem is the possibility of resonant tunneling resulting from the nonlinearity. In a linear disordered system, resonant tunneling is rendered unlikely by the low probability of having two resonant subsystems sufficiently close together to overcome the exponential decay of the wave function between the subsystems; the result is large resonance-free regions in the spectrum, and the subsequent absence of diffusion. A gap in the FSW theory is whether or not there exist special initial conditions which result in resonant tunneling, enhanced by the nonlinearity, between localization sites. An important aspect of our experiment, as will be discussed, is that it can test this possibility. We find that, for a one-dimensional system under the conditions of the FSW paper, the nonlinearity does not weaken the Anderson localization, as predicted by FSW, and that even under conditions which favor nonlinear enhanced tunneling the Anderson localization persists. Our studies involve a range of amplitude of nearly 3 orders of magnitude, up to the point where the system becomes strongly chaotic. The Anderson localization appears to be even stronger at the largest amplitudes. It should be noted that our results do not contradict the other theory papers [3-7] since only the FSW and similar theories correspond to our experimental situation, as described below.

Our experimental system is quite straightforward. The one-dimensional wave medium consists of a steel wire of mass per unit length $\mu=2\times10^{-3}$ g/cm stretched to a tension T_0, so that the speed of low-amplitude transverse waves is $c_0=(T_0/\mu)^{1/2}=400$ m/s. Transverse waves are generated by an electromechanical actuator at one end of the wire. The disordered potential field consists of a sequence of small weights of mass $m=0.12$ g randomly positioned along the wire; the average spacing of the masses is $a=20$ cm, and the positions deviate randomly from periodic lattice positions within a limit of $0.02a$. The masses accurately simulate a Kronig-Penny potential field consisting of a series of delta functions with strength $m\omega^2/T_0$, where ω is the temporal frequency of the transverse waves on the wire. For small amplitude transverse waves, this potential field is found to produce Anderson localized eigenstates with localization lengths on the order of $6a$ [11]. The vibration field of the wire-mass system is measured with an electrodynamic transducer which can be moved along a track running parallel to the wire, recording the amplitude and phase of the vibration of the wire as a function of position. In the experiments reported here, the frequencies were in the neighborhood of what would have been the second transmission band if the system had been periodic; that is, the frequencies were such that approximately one-half wavelength fit between the masses. The losses in the system were quite small; the resonances of the system at low amplitudes had quality factors of ~1500. The method of making measurements was as follows: An amplitude for the drive actuator was selected, the receive transducer was left in one position, the frequency of the drive was slowly swept, and the spectral response of the system was recorded. Using the spectral response, particular frequencies, corresponding to Anderson localized states at low amplitudes, could be selected, and the receive transducer could be translated along the wire, recording the wave field for the selected frequency. The measurements were repeated for a sequence of increasing drive amplitudes, revealing the effects of the nonlinearity of the system.

The nature of the nonlinearity in our system is one of the important aspects of the experiment. To derive the nonlinear equation governing our system, one first considers how the experimental situation is established. One begins with an unstretched wire of length L_0, then applies the tension T_0 so that the wire stretches to a length $L=L_0+\Delta L$, and then adds the masses. The straight wire of length L is the equilibrium configuration for the system. For infinitesimal transverse displacements from equilibrium the equation of motion for the wire is

$$\mu\frac{\partial^2\Psi}{\partial t^2}-T_0\frac{\partial^2\Psi}{\partial x^2}=0\,, \tag{1}$$

where $\Psi(x,t)$ is the transverse displacement field of the wire. If the wire had a finite transverse displacement, then the arclength of the wire would be greater than L, and the tension in the wire would increase. Any local increase in tension in the wire would travel as a longitudinal sound wave in the wire. Because the speed of longitudinal sound in the steel wire is greater than the speed of the transverse waves in the wire, any change in the tension due to a local transverse displacement produces a virtually instantaneous change in the overall tension of the wire. A good approximation for the net tension T in the wire simply involves the change in arclength for the entire wire:

$$T=T_0+\frac{T_0}{\Delta L/L}\frac{1}{L}\left\{\int_0^L\left[1+\left(\frac{\partial\Psi}{\partial x}\right)^2\right]^{1/2}dx-L\right\}. \tag{2}$$

$T_0/(\Delta L/L)$ is an experimentally accessible expression for the Young's modulus of the wire. A more rigorous derivation of Eq. (2) may be found in Morse and Ingard [12]. Equation (2) may be expanded to first order in $(\partial\Psi/\partial x)^2$ and used to replace T_0 in Eq. (1) to yield the nonlinear equation of motion

$$\mu\frac{\partial^2\Psi}{\partial t^2}-T_0\left[1+\frac{1}{2\Delta L}\int_0^L\left(\frac{\partial\Psi}{\partial x}\right)^2dx\right]\frac{\partial^2\Psi}{\partial x^2}=0\,. \tag{3}$$

Because the nonlinear term involves an integral of the displacement field, the nonlinearity is nonlocal, and this increases the possibility of having nonlinear enhanced tunneling between two localization sites. That is, a large-amplitude transverse displacement at one localization site will modulate the tension in the entire wire at twice the eigenstate frequency. This modulated tension may then parametrically excite a response at a distant localization site. The eigenfrequency (at low amplitude) of the distant site may even differ somewhat from that of the original site. The reason is that the finite-amplitude displacement also increases the effective static tension of the wire, so that lines in the spectral response are bent toward higher frequencies, and may be bent over the top of one another [13]. In this case states with different frequencies at low amplitude may be excited concurrently at the same frequency at finite amplitude. The initial conditions of the experiment (as relevant to the FSW theory) are arbitrary, depending on the state of the system prior to adjusting the frequency of the drive.

With the possible nonlinear effects having been discussed, we can now present the actual experimental results. The simplest way to view the results is to examine the spectral response (amplitude at a fixed site as a function of frequency), measured at a distance of about four localization lengths from the drive actuator and normalized by dividing by the drive amplitude, for different drive amplitudes. If the system were strictly linear, then the normalized response would not change. If the Anderson localization is weakened by the nonlinearity, then, as the drive amplitude is increased, the normalized response at the distant site should increase.

Our experimental results are presented in Fig. 1, which shows a sequence of normalized spectral response plots for a sequence of increasing drive amplitudes. The drive amplitude, expressed as the amplitude of the electrical signal applied to the drive actuator in volts, is shown in the left column of numbers in Fig. 1. Below the lowest amplitude in Fig. 1, the spectral response shows little variation, but in the sequence of increasing drive amplitudes shown, the spectral response shows some change. For some of the peaks in the spectrum, for example, the one indicated by the arrow in Fig. 1, the normalized response increases with increasing drive amplitude, suggesting that there might be some weakening of the Anderson localization. However, this effect does not seem to persist to the highest drive levels. Furthermore, an examination of the wave fields for the peaks which increase indicates that the effect is due to the growth in amplitude of sections of wire between a few masses only. Figure 2 shows an example of one such wave field whose peak increased with increasing drive amplitude. Figure 2(a) is the wave field (wave amplitude, normalized with the drive amplitude, versus position, with the drive to the left in the figure) for a drive of 0.01 V, and Fig. 2(b) is the wave field for a drive of 0.50 V. While the normalized amplitudes of a few sections have increased, the Anderson localization has not changed significantly. It should be noted that for the wave field in Fig. 2, and in all of the measured wave fields, there was no significant harmonic generation observable.

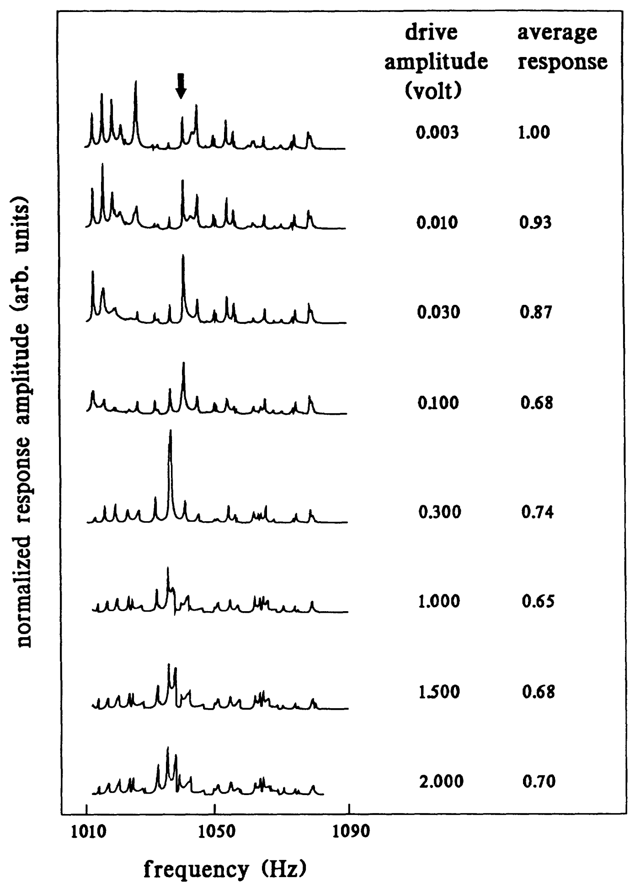

FIG. 1. Normalized spectral response for a sequence of drive amplitudes. The left column of numbers displays the drive amplitude, expressed as the amplitude of the electrical signal applied to the drive actuator in volts. The right column of numbers presents the "average response," defined as the integral of the normalized spectral response over the entire frequency band and normalized to the value at the lowest drive level (0.003 V). The arrow indicates a state whose normalized amplitude increases with drive amplitude, but the effect does not persist.

None of the wave fields at any frequency which was measured showed any significant reduction of Anderson localization, in accord with the FSW theory. Furthermore, an examination of wave fields most likely to show nonlinear enhanced tunneling (i.e., states at nearly the same frequency localized at different sites) gave no evidence of enhanced tunneling. The highest drive amplitude in our measurements corresponded to a nonlinear shift in the eigenfrequencies by as much as 15% of the bandwidth (quite large by acoustic standards). Proceeding to higher drive amplitude was prevented by the onset of strong chaos in the system; because the Anderson localization concentrates the wave energy in a limited region, the state may act like a simple oscillator which might easily go chaotic.

Returning to Fig. 1, it can be seen that most of the normalized response seems to decrease slightly with increas-

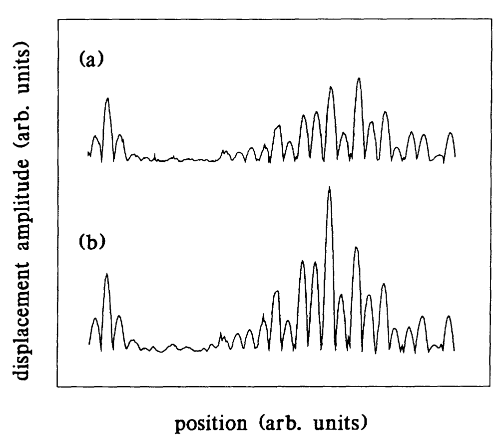

FIG. 2. The wave amplitude, normalized with the drive amplitude, vs position, with the drive actuator to the left. (a) The wave field for a drive amplitude of 0.01 V. (b) The wave field for a drive amplitude of 0.50 V.

ing drive level, indicating stronger localization. A quantitative measure of this effect may be found with an "average response," defined as the integral of the normalized spectral response over the entire frequency band. The results for each drive level, normalized to the value at the lowest drive level in Fig. 1, are presented in the right column of numbers in Fig. 1. The decrease in the average response of about 30% with increasing drive amplitude suggests that the Anderson localization is slightly enhanced by the nonlinearity. One might imagine that the nonlinearity causes an Anderson localized state not to parametrically excite a distant site, but rather to "dig a deeper hole" for itself.

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