Eigenvalue Spectrum, Density of States, and Eigenfunctions in a Two-Dimensional Quasicrystal

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(Received 20 October 1988)

In this Letter we report the experimental determination of the eigenvalue spectrum, density of states, and individual eigenfunctions in an acoustic system analogous to a two-dimensional Schrödinger equation with a quasiperiodic (Penrose tile) potential. The results show features unique to the quasiperiodic symmetry, such as the appearance of gaps and bands with widths which are in the ratio of the golden mean, $(\sqrt{5}+1)/2$.

Since the discovery of aluminum alloys with long-range fivefold rotational symmetry by Shechtman et al.,[1] there has been increased interest in quasiperiodic symmetry. While there is now a well established paradigm of "quasicrystallography"[2,3] concerning the possible quasiperiodic structures, there is considerably less knowledge about how quasiperiodic symmetry affects the various physical properties of a system. For example, a fundamental question is that given a (Schrödinger) wave equation with a quasiperiodic potential field, how is the quasiperiodic symmetry manifest in the eigenvalue spectrum and the eigenfunctions? For one-dimensional (1D) quasiperiodic systems, rigorous theorems[4-7] have been derived which answer such questions: However, in two and three dimensions there are as yet no accepted theorems,[8] and, as will be discussed below, numerical calculations[9-11] have not addressed aspects of the wave equation. In this paper we report measurements with an acoustic experiment which simulates a two-dimensional (2D) Schrödinger equation with a quasiperiodic potential, and which directly determines the eigenvalues, density of states, and eigenfunctions. The results show features which seem unique to the 2D quasiperiodic symmetry, such as the appearance of gaps and bands in the eigenvalue spectrum with widths which are in the ratio of the golden mean, $(\sqrt{5}+1)/2$. It is hoped that these results will provide some insight for developing theories for 2D quasiperiodic wave mechanical systems; a suggestion for extending numerical calculations to search for new effects is already apparent.

For 1D quasiperiodic systems, general theories exist; these have been reviewed by Simon.[4] Renormalization-group and dynamic mapping techniques have been introduced by Kohmoto, Kadonoff, and Tang[5] and by Ostlund et al.[6] Since the discovery of the aluminum alloy quasicrystals,[1] there has been considerably more work with renormalization-group techniques and numerical calculations.[7,12] Some special properties of 1D quasiperiodic systems are the following: (1) the eigenvalue spectrum is a Cantor set, (2) there may exist a mobility edge and a metal-insulator transition, and (3) the states may be extended, localized, or critical.

By contrast little is known about the consequences of quasiperiodic symmetry in two and three dimensions; the transfer-matrix method and the renormalization-group approach seem to be inapplicable. There are some known theorems dealing with structure,[13] such as those involving inflation rules and Conway's theorem, which state that a given local pattern of some diameter will be repeated within a distance of two diameters, and most probably within one diameter. In the absence of a "quasi-Bloch theorem,"[8] progress in understanding the consequences of 2D quasiperiodic (Penrose tile[13]) symmetry has relied mostly on numerical calculations.[9-11] However, these calculations have found no features which reflect the quasiperiodic symmetry.

Despite the notable efforts in searching for unique consequences of 2D quasiperiodic symmetry, existing studies do not treat the problem which we wish to address, namely, the properties of a wave equation with a quasiperiodic potential. All of the current theoretical research has dealt with a hopping Hamiltonian involving a matrix embodying the quasiperiodic topology, but having all nonzero matrix elements identical. For an actual wave equation the problem can be reduced to a similar matrix, but the nonzero matrix elements would be complex functions of the eigenvalue, whose determination would involve solving a complicated transcendental equation (an example is presented below). The resulting eigenvalue spectrum would be quite different from the ones found with the existing theoretical models, because now one must contend with the possibility of phase-coherence effects in a system of scatters with a quasiperiodic pattern. Another way of viewing the situation is to note that another length, the wavelength, has entered the problem. The relation between the wavelength and the inflation and pattern repetition properties of the quasiperiodic pattern results in new features in the density of states.

The study of energy eigenstates of the Schrödinger equation through the use of acoustic wave measurements is discussed in the literature[14]; basically, the salient features of the quantum system and the acoustic system can be made mathematically identical. A convenient

description of our acoustic 2D quasiperiodic system as an analog of the Schrödinger equation may be obtained with a "tight-binding" approach, where local oscillators, which when isolated have a single sharp eigenvalue or resonant frequency, are coupled together to produce a spectrum of eigenvalues. While this seems identical to the models used in the numerical studies, there is a crucial difference: In the numerical studies the coupling has been analogous to "massless springs," whereas in our acoustic system the coupling is through a wave medium (a spring with a finite mass) involving a wavelength, phase coherence, etc.

As an example, consider a system where each local oscillator is a mass m on a massless spring with resonant frequency ω_0, and the coupling to nearest neighbors (NN) is through waves in other springs with sound speed c, individual masses m_{NN}, and lengths l_{NN}. Following the notation of Ref. 11, where $\psi(x)$ is the amplitude at a site x and $\psi(x')$ is that of a nearest neighbor, the equations of motion are[15]

$$E\psi(x) = -\sum_{NN} [a'\psi(x') - a\psi(x)], \qquad (1)$$

where in the acoustic case the eigenvalue E is equal to the square of the acoustic resonance frequency, $E = \omega^2$, and

$$a' = \omega_{NN}^2 \left[\frac{\omega l_{NN}}{c} \csc \frac{\omega l_{NN}}{c} \right], \qquad (2)$$

$$a = \frac{\omega_0^2}{z} + \omega_{NN}^2 \left[\frac{\omega l_{NN}}{c} \cot \frac{\omega l_{NN}}{c} \right], \qquad (3)$$

where $\omega_{NN} = (c/l_{NN})(m_{NN}/m)^{1/2}$ and z is the number of nearest neighbors. These equations illustrate the comments made in the earlier paragraph: If the coupling springs were massless, the speed of sound, proportional to $1/\sqrt{m_{NN}}$, would be infinite (ω_{NN} would remain finite), the factors in parentheses in Eqs. (2) and (3) would be unity, and the matrix elements a' and a would become constants, no longer functions of the eigenvalue. This is the model used in existing numerical calculations. However, in order to have a wave equation, c must be finite, and determining the eigenfrequencies ω and a' and a functions of ω involves a complicated transcendental equation.

For the acoustic system to be a precise analog of the Schrödinger equation it must have relatively little damping; i.e., the local oscillators and the coupling mechanism must have a high quality factor (Q). For the local oscillators in our acoustic simulation we use ordinary commercial tuning forks (440 Hz). These have the advantage that they can be mounted by the stem and still maintain a high-Q oscillation. The tuning forks are expoxied in a 2D quasiperiodic pattern into a heavy aluminum plate; the pattern is a standard Penrose tile formed with two rhombuses (fat and skinny) having areas in the
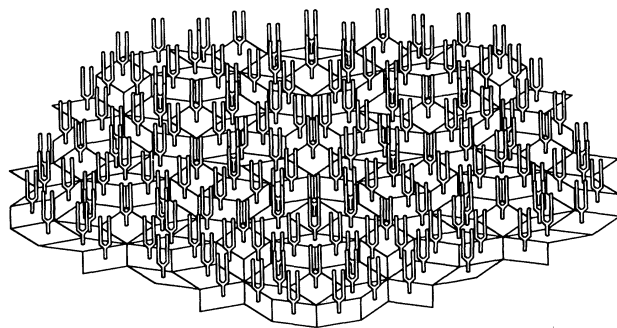

FIG. 1. A schematic drawing of the tuning fork quasicrystal. The tuning forks are mounted at the center of the rhombuses in the Penrose tile, with the two tines oriented in line with the shorter diagonal (for the sake of drafting simplicity, the tuning forks are not drawn with the correct orientation). For the nearest-neighbor coupling, arcs of steel wire (not shown) are spot welded from one tine of a tuning fork to that of a nearest neighbor.

ratio of the golden mean $(\sqrt{5}+1)/2$. An illustration of the tuning fork quasicrystal is shown in Fig. 1. The tuning forks are mounted at the centers of the rhombuses, with the two tines oriented in line with the shorter diagonal. For the nearest-neighbor coupling, arcs of 1-mm-diam steel wire are spot welded from one tine of a tuning fork to that of a nearest neighbor. Other coupling schemes were tested and found to be either too weak, too lossy, or had too low a coupling wave velocity. Using the four sides of each rhombus, four nearest neighbors are identified, and each tine of a tuning fork is coupled to the two nearest tines of the adjacent tuning forks.

With the coupling scheme just described each local oscillator has four nearest neighbors, but the nearest-neighbor length l_{NN} varies in a quasiperiodic pattern. There are four different lengths: 3.7, 5.8, 7.26, and 7.77 cm. From a separate measurement of the speed of sound c in the coupling wires, the phase shifts $\omega l_{NN}/c$ [see Eqs. (2) and (3)], within the total band of frequencies of our experiment (385 to 525 Hz), are found to range from $\sim 2\pi/3$ to $\sim 2\pi/6$. Thus in rings of several tuning forks there is the opportunity for constructive or destructive interference. The variation of this interference with frequency undoubtedly gives rise to the structure in the observed eigenvalue spectrum; since in the Penrose pattern the golden mean is involved in various length scales, and thus also in the phase shifts, it is conceivable that this ratio should appear in the eigenvalue spectrum.

It should be noted that the tuning forks, with two tines each and 2D displacements and coupling forces, have equations of motion which are more complicated than the example illustrated in Eqs. (1)–(3). However, regardless of the complexity of the local oscillator and the coupling, our system still involves the wave equation with 2D quasiperiodic symmetry.

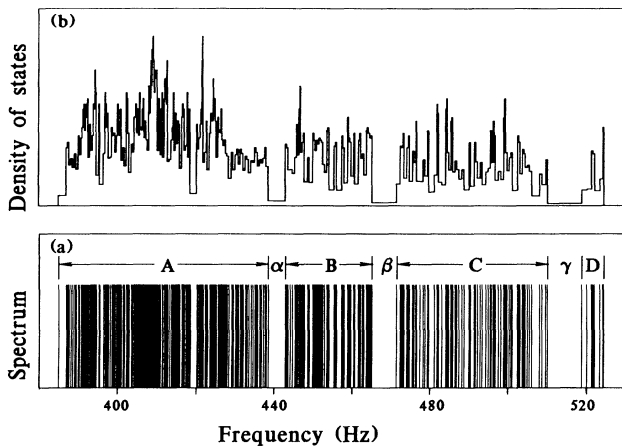In order to drive the oscillations of the coupled tuning

FIG. 2. (a) The eigenvalue spectrum of the tuning fork quasicrystal. It is determined as a composite of the resonant spectra from twenty different positions in the Penrose pattern, and shows the gaps and bands whose widths are in the ratio of the golden mean $(\sqrt{5}+1)/2$. (b) The density of states as the inverse of the difference in frequency for neighboring eigenvalues.

fork system, an electromagnet is positioned near one tine of the array, and an ac current is passed through the electromagnet. The response of the system is monitored with four electrodynamic transducers (electric guitar pickups) positioned next to random tines in the array. By sweeping the frequency of the drive electromagnet, the resonant response of the system is detected with the pickup transducers; the resonant frequencies correspond to the eigenvalues of the quasiperiodic system. The eigenvalue spectrum, determined as a composite of the resonant spectra from twenty different positions in the Penrose pattern, is presented in Fig. 2(a). This spectrum shows gaps and bands whose widths are in the ratio of the golden mean, $\tau = (\sqrt{5}+1)/2$. Referring to Fig. 2(a), we have $\beta/\alpha = \tau$, $\gamma/\beta = \tau$, $C/B = \tau$, $A/B = \tau^2$, $B/D = \tau^3$, and other ratios involving combinations of these, with an average deviation of $\pm 5\%$. The density of states, determined as the inverse of the difference in frequency for neighboring eigenvalues, is shown in Fig. 2(b).

Measuring an eigenfunction involves driving the tuning fork array at one of its eigenfrequencies and recording the motion of each of the 300 tines. It would have been a prohibitive task to instrument and calibrate each tine (including two-dimensional movement). Instead, a small mirror is placed at the end of each tine, and the tuning fork array is mounted on one wall of a dark room, with a drawing depicting the mirror image of the Penrose pattern mounted on the opposite wall. The mirrors are adjusted so that a laser beam directed to one of the tines is reflected to the corresponding position of the tine in the image drawing. If a tine is vibrating, then its motion (amplitude and polarization) will be reproduced in the motion of the laser spot in the image drawing, but
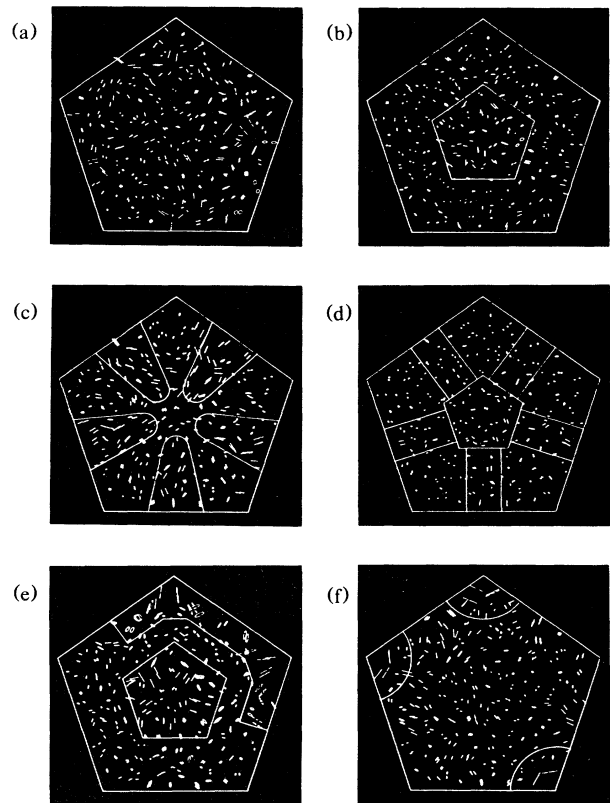

FIG. 3. Several eigenfunctions of the tuning quasiperiodic system. They show the motion (both amplitude and polarization) of the system (note that the longer the laser spot, the larger the amplitude). In (a)-(d) energy distributions have fivefold rotational symmetry, in (e) and (f) the distributions have a symmetry of a mirror plane.

with an amplified displacement resulting from the optical lever arm provided by the laser beam traversing the room. By scanning the laser beam over the entire tuning fork array, the motion of all the tines may be observed in the motion of the laser spots in the image drawing. A time-exposed photograph of the moving laser spots taken during the scan records lines, ellipses, or circles of various amplitudes indicating the motion of each of the tuning forks; this is essentially a photograph of the eigenfunction at that particular eigenfrequency.

Several eigenfunctions are shown in Fig. 3. It should be noted that the tuning fork quasicrystal is not perfect; there are defects arising primarily from unavoidable variations in the spot welds between the tines and the coupling wires. As a consequence of the adiabatic theorem for small perturbations, the defects have a minor effect on the eigenvalue spectrum, so that the observations concerning the gaps and bands are still valid. However, the defects may have a much greater effect on the eigenfunctions; furthermore, it was not possible to align the mirrors and identify the individual tines exactly

in the eigenfunction photographs. As a consequence the photographs are used only to obtain a qualitative idea of the nature of the eigenfunctions. Figure 3 illustrates some of the basic patterns which were found. While the original photographs have better resolution than the reproductions in Fig. 3, some scrutiny was necessary to discern the patterns; in Fig. 3 we have drawn lines to indicate the general areas containing the larger amplitudes. In 3(a) energy is uniformly distributed; in 3(b) energy is concentrated in the center; in 3(c) energy is concentrated in the five symmetric arms; in 3(d) energy is both concentrated in the center and in the arms; in 3(e) energy is localized in the center and partly at the edges; and in 3(f) energy is localized in three of the five corners. The lack of fivefold symmetry in 3(e) and 3(f) may be due to a combination of degenerate eigenstates. With these results we verify that states with fivefold symmetry can be observed and, at least for this finite-size system, highly localized, intermediately localized, and nonlocalized states exist. It should be noted that for a quasiperiodic system with defects, the occurrence of gaps in the eigenvalue spectrum creates the possibility of more highly localized states at the band edges, and this will have significant consequences for transport in real icosahedral materials.[16]

[1]D. Schechtman et al., Phys. Rev. Lett. 53, 1951 (1984).

[2]D. Levine and P. J. Steinhardt, Phys. Rev. B 34, 596 (1986); J. E. S. Socolar and P. J. Steinhardt, Phys. Rev. B 34, 617 (1986); D. S. Rokhsar, D. C. Wright, and D. N. Mermin, Phys. Rev. B 37, 8145 (1988).

[3]G. Y. Onoda, P. J. Steinhardt, D. P. DiVincenzo, and J. E. S. Socolar, Phys. Rev. Lett. 60, 2653 (1988).

[4]B. Simon, Adv. Appl. Math. 3, 463 (1982).

[5]M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. 50, 1870 (1983).

[6]S. Ostlund et al., Phys. Rev. Lett. 50, 1873 (1983); S. Ostlund and R. Pandit, Phys. Rev. B 29, 1394 (1984).

[7]J. M. Luck and D. Petritis, J. Stat. Phys. 42, 289 (1986); J. P. Lu, T. Odagaki, and J. L. Birman, Phys. Rev. B 33, 4809 (1986); M. Kohmoto and J. R. Banavar, Phys. Rev. B 34, 563 (1986); Q. Niu and F. Nori, Phys. Rev. Lett. 57, 2057 (1986); M. Kohmoto, B. Sutherland, and C. Tang, Phys. Rev. B 35, 1020 (1987); G. Gumbs and M. K. Ali, Phys. Rev. Lett. 60, 1081 (1988); M. Luban, J. H. Luscombe, and S. Kim, Phys. Rev. Lett. 60, 2689 (1988).

[8]A theory has been proposed by J. P. Lu and J. L. Birman [Phys. Rev. B 36, 4471 (1987)], but D. Thouless and D. P. DiVincenzo (private communication) have indicated that the theory may not be generally applicable.

[9]T. C. Choy, Phys. Rev. Lett. 55, 2915 (1985); T. Odagaki and D. Nguyen, Phys. Rev. B 33, 2184 (1986).

[10]M. Marcus, Phys. Rev. B 34, 5981 (1986).

[11]M. Kohmoto and B. Sutherland, Phys. Rev. B 34, 3849 (1986); Phys. Rev. Lett. 56, 2740 (1986); H. Tsunetsugu, T. Fujiwara, K. Ueda, and T. Tokihiro, J. Phys. Soc. Jpn. 55, 1420 (1986).

[12]M. Luban and J. H. Luscombe, Phys. Rev. B 35, 9045 (1987).

[13]R. Penrose, Bull. Inst. Math. Appl. 10, 266 (1974); M. Gardner, Sci. Am. 236, 110 (1977).

[14]S. He and J. D. Maynard, Phys. Rev. Lett. 57, 3171 (1986).

[15]L. E. Kinsler and A. R. Frey, Fundamentals of Acoustics (Wiley, New York, 1962).

[16]J. B. Sokoloff, Phys. Rev. Lett. 57, 2223 (1986).